ELSEVIER
Journal of Pure and Applied Algebra 120 (1997) 221–233

JOURNAL OF
PURE AND
APPLIED ALGEBRA

Varieties of simplicial groupoids I: Crossed complexes

P.J. Ehlers, T. Porter *

School of Mathematics, University College of North Wales, Dean Street, Bangor, Gwynedd, LL57 1UT, UK

Communicated by C.A. Weibel; received 7 December 1994; revised 28 June 1995

Abstract

It is usual to use algebraic models for homotopy types. Simplicial groupoids provide such a model. Other partial models include the crossed complexes of Brown and Higgins. In this paper, the simplicial groupoids that correspond to crossed complexes are shown to form a variety within the category of all simplicial groupoids and the corresponding verbal subgroupoid is identified. © 1997 Elsevier Science B.V.

1991 Math. Subj. Class.: 55P15, 55U35, 18G30, 18G55

0. Introduction

Algebraic topology aims to translate topological structure to algebraic structure. Homotopy types, via their algebraic models, thus become amenable to algebraic manipulation (for instance localization), which makes the information they contain more accessible.

The theory of varieties provides a useful set of tools in algebra and, in particular, in group theory, which are similar in many ways to those of localization. Any group variety leads to a set of 'equational laws' satisfied by the groups in it and verbal subgroup functors (such as the commutator subgroup) measure deviation from membership of the variety.

The main aim of this paper is to give an example of a variety in a category of algebraic models for homotopy types, and to describe the corresponding equational laws by means of a verbal subgroup construction.

* Corresponding author.

PII S0022-4049(96)00038-2

The algebraic model we shall consider is that of simplicial groupoids. These model all homotopy types completely [11]. Our variety will be a subcategory consisting of groupoid T-complexes. These form a category equivalent to the category of crossed complexes. Crossed complexes have been extensively studied by Brown and Higgins [3] and can either be viewed as a complete algebraic model for a restricted class of homotopy types or as partial models of the homotopy types of all spaces. Their advantage is that they contain information on the fundamental groupoid of the space and thus are 'slightly non-abelian'.

The main points of this paper are thus:

(i) to prove that *Gpd–T*, the category of groupoid T-complexes is a variety (epireflective subcategory) in the category, $SGpds_*$, of simplicial groupoids with constant object of objects;

(ii) to identify the verbal subgroupoid corresponding to the variety of groupoid T-complexes.

1. Preliminaries

1.1. Simplicial groupoids

We refer the reader to Curtis [9] for a brief overview of the theory of simplicial groups. We will need extensions of some of this theory to simplicial groupoids or to be more exact to simplicial groupoids whose simplicial set of objects, or identities, is constant. Results on such simplicial groupoids are mostly parallel to, and extensions of, the corresponding group versions and are proved in a similar way. The full subcategory $SGpds_*$ of the category $SGpds$, of simplicial groupoids is defined by the condition:

$$G \in SGpds_* \Leftrightarrow Ob(G) = X \text{ for some set } X.$$

If some X is given and fixed for the duration of some argument, we may also consider a subcategory, $SGpds/X$, of $SGpds_*$ in which all objects have $Ob(G) = X$ and $f : G \to H$ is in $SGpds/X$ if and only if the object function, $Ob(f)$, of f is the identity map of X. If X is a singleton set then $SGpds/X$ is equivalent to the category of simplicial groups.

1.2. The Dwyer–Kan path groupoid construction

Let K be a simplicial set, with vertex set K_0. We define $(GK)_n$ to be the groupoid with object set $\{\overline{x} : x \in K_0\}$ and morphisms generated by the 'edges'

$$\overline{y} : \overline{d_1 d_2 \ldots d_{n+1} y} \to \overline{d_0 d_2 \ldots d_{n+1} y}$$

for all $y \in K_{n+1}$, with relations

$$\overline{s_0 z} = id_{\overline{d_1 d_2 \dots d_n z}}$$

for all $z \in K_n$.

These groupoids are free, as the relations have the effect of deleting certain generating edges, thus to define face and degeneracy maps between them, it is sufficient to define them on generating edges:

- for $i \geq 0$, $\sigma_i : (GK)_n \to (GK)_{n+1}$ is given by $\sigma_i \overline{x} = \overline{s_{i+1} x}$,
- for $i > 0$, $\delta_i : (GK)_{n+1} \to (GK)_n$ is given by $\delta_i \overline{x} = \overline{d_{i+1} x}$, whilst $\delta_0 : (GK)_{n+1} \to (GK)_n$ is given by $\delta_0 \overline{x} = (\overline{d_1 x})(\overline{d_0 x})^{-1}$.

Dwyer and Kan proved in [11] that G has an adjoint $\overline{W}$, and that the counit of the $G, \overline{W}$ adjunction is a weak equivalence $K \to \overline{W} GK$, thus simplicial groupoids model all homotopy types. In a simplicial groupoid G, let $D_n(G)$ be the subgroupoid of G_n generated by the degenerate elements. (Usually we will write D_n instead of $D_n(G)$ if the particular groupoid, G, is clear.)

Proposition 1.1. *A simplicial groupoid is a Kan complex and furthermore, any box in G_{n-1} has a filler in D_n.*

1.3. The homotopy theory of a simplicial groupoid

The homotopy theory of simplicial groupoids is parallel to that of simplicial groups. Let G be a simplicial groupoid, then by its Moore complex we mean the chain complex (NG, ∂) of groupoids defined by

$$(NG)_n = \bigcap_{i=1}^{n} Ker d_i^n$$

with $\partial_n : (NG)_n \to (NG)_{n-1}$ being given by the restriction of d_0^n to $(NG)_n$. We note that as the face (and degeneracy) maps of G are the identity on objects for $n \geq 1$, each $Ker d_i^n$ (and hence the intersection $(NG)_n$) is a totally disconnected, normal, wide subgroupoid of G_n, i.e. is the disjoint union of the vertex groups $Ker d_i^n(a)$, $a \in Ob(G)$. In particular $\partial_{n-1}\partial_n$ maps $(NG)_n$ to the discrete groupoid on $Ob(G)$, so that (NG, ∂) is, indeed, a chain complex of groupoids over $Ob(G)$. Thus all but $(NG)_0$ of the groupoids concerned are totally disconnected, i.e. are disjoint unions (coproducts) of groups as groupoids, and $(NG)_0 \cong G_0$. This base groupoid G_0 acts on all the $(NG)_n$ by ${}^h g = ((s_0)^n h) g ((s_0)^n h)^{-1}$ and similarly one checks that $\partial_n (NG)_n$ is normal in $(NG)_{n-1}$.

Using this observation it is easy

(a) to extend the analysis of the 'hypercrossed complex structure' of (NG, ∂) from the reduced case of simplicial groups to this wider context of simplicial groupoids.

(b) to prove a groupoid version of the Carrasco-Cegarra theorem [6], which shows the categories of hypercrossed complexes and simplicial groups to be equivalent. We will not prove this theorem here as the necessary extensions of their results are routine given the extensions of notions of an action and a semidirect product from groups to groupoids.

1.4. Crossed complexes

The theory of crossed complexes can be found mostly in the work of Brown and Higgins (see bibliography for a selection of references.) The following definition can be found in [4] but with a shift in dimension.

A crossed complex C, over a groupoid, is a sequence

$$\cdots \to C_{n+1} \to C_n \to \cdots \to C_2 \to C_1 \to C_0 \rightrightarrows ObC$$

where

(i) $C_0 \rightrightarrows ObC$ is a groupoid with object set $Ob(C)$ (we write $C_0(a)$ for $C_0(a,a)$, $a \in ObC$);

(ii) C_n is a family of groups $\{C_n(a)\}_{a\in ObC}$ for $n \geq 1$ (and hence is also a groupoid over ObC);

(iii) $C_n(a)$ is abelian for $n \geq 2$, $a \in ObC$;

(iv) C_0 acts on C_n, on the left, for all $n \geq 1$, by $(h,g) \mapsto {}^hg$, where if $g \in C_n(a)$ and $h \in C_0(b,a), {}^hg \in C_n(b)$;

(v) the ∂_n are all groupoid morphisms which preserve the action;

(vi) if $x, y \in C_1(a)$, then ${}^{\partial_0 x}y = xyx^{-1}$ and $\partial_0 C_1(a)$ acts trivally on $C_n(b)$ for $n \geq 2$ and all $a, b \in ObC$;

(vii) $\partial_{n-1}\partial_n$ is trivial for $n \geq 1$.

There is an obvious notion of morphism of crossed complexes, giving a category, *Crs*.

1.5. The semidirect decomposition of simplicial groupoid

The basic idea behind this is to be found in Conduché [7]. A detailed analysis of it in the case of a simplicial group is in Carrasco and Cegarra [6]. The decomposition is based on the observation that in a simplicial group, $d_n^n : G_n \to G_{n-1}$ is a split epimorphism, split by s_{n-1}^{n-1}, and as a consequence

$$G_n \cong Kerd_n^n \rtimes s_{n-1}^{n-1}G_{n-1}.$$

One obtains a semidirect decomposition of G_n by iterating this product not only on G_{n-1}, but also on $Kerd_n^n$. (The kernel of the last face map $Kerd_{last}$ is the kernel of a simplicial group epimorphism from $DecG$ to G (for $DecG$, see [13]), hence is a simplicial group in its own right.)

Using the notion of semidirect product for groupoids, one easily adapts this to the case of G in $SGpds_*$:

Proposition 1.2. *Given any simplicial groupoid, G, the groupoid of n-simplices G_n satisfies*

$$G_n \cong (\ldots(NG_n \rtimes s_0NG_{n-1}) \rtimes \cdots \rtimes s_{n-2}\ldots s_0NG_1)$$
$$\rtimes(\ldots(s_{n-1}NG_{n-1} \rtimes s_{n-1}s_0NG_{n-2}) \rtimes \cdots \rtimes s_{n-1}\ldots s_0NG_0).$$

1.6. Simplicial T-complexes

We recall the following definition due to Dakin [10].

A T-complex (K,T) is a pair where K is a simplicial set and $T=(T_n)_{n\geq 1}$ is a graded subset of $(K_n)_{n\geq 1}$. The elements of the T_n are called thin elements. This data is assumed to satisfy the axioms:

T.1. Every degenerate element is thin.

T.2. Every box has a unique thin filler.

T.3. The thin filler of a thin box has a thin lid.

By a box in K_n, we mean a set of elements $x_0,x_1,\dots,x_{i-1},x_{i+1},\dots x_n$ such that $x_j\in K_{n-1}$ $(j\neq i)$ and $d_kx_j=d_{j-1}x_k$ for $j>k$, $j,k\neq i$; a 'filler' for such a box is an element $y\in K_n$ such that $d_jy=x_j$ for $j\neq i$, the 'missing face' d_iy is then called the 'lid'.

Ashley [1] proved that the category of T-complexes is equivalent to that of crossed complexes (over groupoids). He also introduced the notion of a group T-complex. The definition of groupoid T-complexes and the corresponding results are the obvious groupoid versions of his results. A groupoid T-complex (G,T) is a T-complex where G is in $SGpds_*$, each T_n is a subgroupoid of the corresponding G_n and the underlying pair forms a T-complex.

Proposition 1.3. *If (G,T) is a groupoid T-complex, then $T_n=D_n$, the subgroupoid generated by degenerate elements.*

Proposition 1.4. *If G is a simplicial groupoid then (G,D) is a groupoid T-complex if and only if $D\cap NG$ is trivial, i.e. consists only of identities.*

Proof. See [1] for the group case. □

This gives a purely algebraic criterion for G to be a groupoid T-complex.

2. From simplicial groupoids to crossed complexes

First some notation, we will write as above

$$C(G)_n=\frac{NG_n}{(NG_n\cap D_n)d_0(NG_{n+1}\cap D_{n+1})}$$

If $x\in NG_n$, $\bar{x}$ will denote the corresponding element of $C(G)_n$. The map

$$\partial : C(G)_n\to C(G)_{n-1}$$

will be induced by d_0.

We first check that this makes sense.

Lemma 2.1. *The subgroupoid* $(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})$ *is normal in* G_n.

Proof. The proof is easy, so is omitted. □

Proposition 2.2. *Let G be a simplicial groupoid in* $SGpds_*$, *then defining*

$$C(G)_n = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})}$$

with

$$\partial_n(\bar{z}) = \overline{d_0 z},$$

yields a crossed complex, $(C(G), \partial)$ *over a groupoid.*

Proof. This is a consequence of the theory of hypercrossed complexes of Carrasco and Cegarra, [6]. A short (5 page) direct proof is the subject of the note [12]. □

We note that if G is a groupoid T-complex then $(C(G), \partial) \cong (NG, \partial)$. The question naturally arises as to whether there is a functor $T : SGpds_* \to SGpds_*$, taking values in the subcategory of groupoid T-complexes, and such that $C(G) \cong N(T(G))$. It is clear how to proceed, namely by using the semidirect decomposition of G, replacing each NG_n by $C(G)_n$. In fact it is reasonably easy to construct a functor K 'inverse' to N, $K : Crs \to SGpd_*$ and then we can set $T = KC$. The construction is based on the following elementary observation.

Lemma 2.3. *Let M,N be G-groups (i.e. groups with a G-action), and form* $M \times N$, *their product with the diagonal action of G. Then*

$$(M \times N) \rtimes G \cong M \rtimes (N \rtimes G)$$

where M is considered as a $(N \rtimes G)$*-group via the projection onto G (so* ${}^{(n,g)}m = {}^{g}m$*).*

The proof is omitted as it is easy.

Proposition 2.4 (A Dold–Kan theorem for groupoid T-complexes). *There is a functor* $K : Crs \to SGpds_*$ *so that*

(i) *for each C in Crs, KC is a groupoid T-complex,*

(ii) *if C is in Crs, there is a natural isomorphism*

$$NK(C) \cong C,$$

(iii) *if G is a groupoid T-complex,* $KN(G) \cong G$.

Proof. We first note that there is a chain complex (with one non-abelian groupoid) given by all C_i, $i \geq 1$. From this we construct a simplicial group $K_{\bullet+1}(a)$ for each $a \in Ob(C)$ using the Dold–Kan construction [9]. (Thus for instance

$$K_3(a) = C_3(a) \times s_0C_2(a) \times s_1C_2(a) \times s_1s_0C_1(a).)$$

As there is no action of any group on any other in the chain complex, there is no problem in constructing $K_{\bullet+1}$. Now we set $K(G)_0 = C_0$, and note that it acts on $K_1 = \{C_1(a)\}$. Assuming that $K(G)_{n-1}$ has been defined and an action of $K(G)_{n-1}$ on K_n given, we set

$$K(G)_n = K_n \rtimes s_{n-1}K(G)_{n-1}$$

and, noting that there is an iterated projection onto C_0, make $K(G)_n$ act on K_{n+1} via this projection. It is routine to check that this gives us a simplicial groupoid $K(C)$ with Moore complex isomorphic to C, thus proving (ii). If $C = N(G)$ for a groupoid T-complex, it is clear that $KC \cong G$. We leave the details of this to the reader.

It remains to prove that $KC(G)$ is a groupoid T-complex. For this we note that by repeated use of Lemma 2.3, $K(C) \cong C_n \rtimes (s_0C_{n-1} \rtimes \cdots \rtimes s_{n-1}\dots s_0C_0)$ but $NK(C)_n \cong C_n$, $D_n \cong (s_0C_{n-1} \rtimes \cdots \rtimes s_{n-1}\dots s_0C_0)$ and hence $NK(C)_n \cap D_n$ is trivial as required. □

Various other versions of this result have been proved previously. Nan-Tie [14] proves a Dold–Kan theorem exactly of this form, but with a different definition of groupoid T-complexes; see also [15] The above result (2.4), hence also proves that his definition is equivalent to that given here. The reduced case can be found in [6] and as we noted several of the ideas from their proof are present in the above.

Although we know that $C(G)$ is a quotient of NG, we do not automatically have that $KC(G)$ is a quotient of G. The main result of the argument so far is:

Theorem 2.5. *Let Gpd–T denote the full subcategory of $SGpds_*$ determined by the groupoid T-complexes, then the inclusion of Gpd–T into $SGpds_*$ has a left adjoint $T = KC$ which satisfies $T^2 \cong T$.*

Proof. If G is in $SGpds_*$, then the quotient map $NG \to C(G)$ is compatible with the hypercrossed complex structure of the two Moore complexes (cf. the argument on p. 223 of Carrasco and Cegarra [6]) and hence corresponds to a quotient map

$$G \to T(G).$$

If $f : G \to H$ is any map of simplicial groupoids with H a groupoid T-complex, then we may assume that f is over a fixed base map since otherwise, we can pull H back along the base map of f to reduce to that case. As H is a groupoid T-complex,

$$f_n((NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1}))$$

is trivial, so f factors uniquely via a quotient map to $T(G)$ as required. □

We thus have a composite functor

$$Simp.Sets \to Gpd\text{–}T$$

given by TG.

The groupoids $TG(K)_n$ have been calculated to be $\pi_n(sk_n G, sk_{n-1} G)$ [6] and hence to be $\pi_{n+1}(sk_{n+1}K, sk_n K)$ by a long exact sequence argument. Thus $C(G(K)) \cong \pi(\mid K \mid)$, the crossed complex associated to the filtered geometric realization of K.

Remark. It is worth noting that if $L : A \to Simp.Sets$ is a functor and $K = \mathrm{Colim}\, L$, then (i) $G(K) \cong \mathrm{Colim}\, G(L)$ (ii) $C(G(K)) \cong \mathrm{Colim}\, C(G(L))$ and (iii) $T(G(K)) \cong \mathrm{Colim}\, T(G(L))$ as each of G, C and T are left adjoints and hence preserve colimits.

3. The 'verbal' subgroup(oid)

The reason for the title of this section is that in proving Theorem 2.5 above we obtained a quotient map from each G to the corresponding $T(G)$ which is analogous to the map comparing a group, G, with its '$\mathscr{V}$-ification' for $\mathscr{V}$ a variety of groups. The kernel of such a map is the verbal subgroup of G corresponding to $\mathscr{V}$. Thus a 'verbal subgroup(oid)' would be the natural thing to look for in our setting, giving those words whose vanishing is necessary and sufficient for a simplicial groupoid to be in *Gpd–T*. We have of course, already one description of this verbal subgroupoid $\mathscr{V}(G)$, namely it satisfies

$$N\mathscr{V}(G)_n = (NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1}),$$

however this does not give a useful description of $\mathscr{V}(G)$ as we have little direct knowledge of what $NG_n \cap D_n$, or $d_0(NG_{n+1} \cap D_{n+1})$ looks like.

3.1. Derived modules and relative abelianisations

To analyse $\mathscr{V}(G)$ in more detail, we use a construction from [16], adapted to the groupoid case, and later on a related construction of Brown and Higgins [4].

Let O be a set, then we will call a groupoid with object set, O, simply an O-groupoid. Let G be an O-groupoid and *O–Gpds*, the subcategory of *Gpd* determined by the O-groupoids and the groupoid morphisms between O-groupoids that are the identity on objects. Let *O–Gpds/G* denote the category of O-groupoids over G. This category has finite products given by pullback over G, so we can consider abelian group objects in it. A calculation (going back in essence at least to [2]) shows that $\phi : H \to G$ is an abelian group object in *O–Gpds/G* if and only if $H \cong M \rtimes G$ for some G-module M with ϕ the projection. The construction we will use is the abelianization, i.e. the left adjoint of the inclusion of the category of these abelian group objects into *O–Gpds/G*. Now consider a general

$$(H, \phi) \to (M \rtimes G, pr_G)$$

then

$$f(h) = (f_1(h), \phi(h))$$

for some mapping $f_1 : H \to M$. This f_1 is a ϕ-derivation (again see [4, p. 38]),

$$f_1(hh') = f_1(h) + {}^{\phi(h)}f_1(h').$$

Writing $\mathrm{Der}_\phi(H,M)$ for the set of ϕ-derivations from H to M, we have

$$O\text{–}Gpds/G((H,\phi),(M \rtimes G, pr_G)) \cong \mathrm{Der}_\phi(H,M)$$

and as $(M \rtimes G, pr_G)$ is an abelian group object, $\mathrm{Der}_\phi(H,M)$ naturally has an abelian group structure.

If O is a singleton, that is if we are working with groups throughout, then $Der_\phi(H,M) \cong G\text{–}Mod(D_\phi, M)$, where D_ϕ is the derived module of (H,ϕ) in the sense of [8]. This is well known to be isomorphic to $IH \otimes_H \mathbf{Z}G$ where $IH = ker(\mathbf{Z}H \to \mathbf{Z})$ is the augmentation ideal of the group ring $\mathbf{Z}H$ of H. Brown and Higgins [4, pp. 37–39]), show that in the general case of O-groupoids, a similar construction works yielding a G-Module $\overrightarrow{\mathbf{Z}G}$, a constant G-module $\vec{\mathbf{Z}}$, with value $\mathbf{Z}$, and an augmentation module $\overrightarrow{\mathbf{I}G}$ which is the derived module for the terminal object (G, Id_G) of O-$Gpds$. (A right action is used in [4] but this is easily changed to a left one.) Finally D_ϕ, for a general (H,ϕ), is isomorphic to $\phi_*(\overrightarrow{\mathbf{I}H})$, the G-module induced from $\overrightarrow{\mathbf{I}H}$ along $\phi : H \to G$. This ϕ_* functor from H-modules to G-modules is a left additive Kan extension along $\phi : H \to G$, so has a description very much like that of $- \otimes_H \mathbf{Z}G$ of which it is a generalization. Thus

$$\begin{aligned} &O\text{–}Gpds/_G((H,\phi),(M \rtimes G, pr_G)) \\ &\quad \cong Ab(O\text{–}Gpds/_G)((D_\phi \rtimes G, pr_G),(M \rtimes G, pr_G) \cong G\text{–}Mod(D_\phi, M), \end{aligned}$$

i.e. the "abelianization" of (H,ϕ), or free abelian group object on (H,ϕ), is $(D_\phi \rtimes G, pr)$. We next turn to exactness properties of this derived module construction.

Lemma 3.1. *Given a short exact sequence*

$$1 \to N \to H \to Q \to 1$$

in O–$Gpds$, there is a short exact sequence of Q-modules,

$$0 \to N^{\mathrm{Ab}} \to D_p \to \overrightarrow{\mathbf{I}Q} \to O$$

where N^{Ab} is obtained by abelianizing all the groups of N, which being a kernel in O–$Gpds$ is a disjoint union of the groups $N(a)$ for $a \in O$.

This follows from Proposition 3.1 of Brown–Higgins [4, p. 46]. In fact we only use Lemma 3.1 in the simpler case where p is split, in which case the module sequence is also split:

$$O \to N \to H \underset{p}{\overset{s}{\leftrightarrows}} Q \to O$$

yields

$$O \to N^{Ab} \to D_p \underset{s'}{\overset{p'}{\rightleftarrows}} \overrightarrow{\mathbf{I}Q} \to O$$

so we have an isomorphism

$$D_p \cong N^{Ab} \oplus s'_*(\overrightarrow{\mathbf{I}Q}).$$

Now assume in addition this all happens in O–$Gpds/G$ so that the structure map from N to G is trivial. Then we have

$$\begin{aligned} D_\psi &= \psi_*(\overrightarrow{\mathbf{I}Q}),\\ D_\phi &= \psi_* p_*(\overrightarrow{\mathbf{I}H}) = \psi_*(D_p) \cong D_0 \oplus s_*(D_\psi),\\ D_0 &= \psi_*(N^{Ab}). \end{aligned}$$

If ψ is a quotient map, then $\psi_*(N^{Ab})$ is constructed by killing the action of $Ker\psi$, i.e.

$$\psi_*(N^{Ab}) \cong \frac{N}{[N,N][N,s(Ker\psi)]}$$

As both N and $Ker\psi$ are disconnected, this is effectively the classical construction at each object in O. We have

Lemma 3.2. *In the above situation, there is a natural isomorphism*

$$D_\phi \cong \left(\frac{N}{[N,N][N,s(Ker\psi)]}\right) \oplus s'(D_\psi).$$

Now consider a simplicial groupoid over G (e.g. a resolution in O–Gpd for G or the augmentation map $H \to \pi_0 H$ for a simplicial groupoid $G = \pi_0(H)$.) We write $\phi_0 : H_0 \to G, \phi_1 = \phi_0 d_0 = \phi_0 d_1$ and so on, so that $\phi_n : H_n \to G$ is the structure map of the groupoid of n-simplices, $\phi_n d_j = \phi_{n+1}$ for all i, $0 \le i \le n$, and similarly for the s_j, $\phi_n s_j = \phi_{n-1}$. We will also write this as $\phi_\bullet : H_\bullet \to G$, thinking of G as being $K(G,0)$, the constant simplicial groupoid on G.

On applying the abelianisation functor (over G) as before, this yields $D(\phi)_\bullet \rtimes G$ over G which in dimension n is $D_{\phi_n} \rtimes G$. We now restrict to the case of H being a simplicial groupoid, with $G = \pi_0(H_\bullet)$ and ϕ_0 being the standard quotient morphism. In this case, $Ker\phi_0$ is $d_0 NH_1$ so we obtain

$$\begin{aligned} D(\phi)_0 &\cong \phi_{0*}(\overrightarrow{\mathbf{I}H}),\\ D(\phi)_1 &\cong \frac{NH_1}{[NH_1,NH_1][NH_1,s_0 d_0 NH_1]} \oplus s_0(D_{\phi_0}),\\ D(\phi)_2 &\cong \frac{NH_2}{[NH_2,NH_2][NH_2,s_0 NH_1][NH_2,s_1 NH_1][NH_2,s_1 s_0 d_0 NH_1]}\\ &\quad \oplus s_0(ND(\phi)_1) \oplus s_1(D(\phi)_1), \end{aligned}$$

and so on. The semidirect decomposition of H_n yields the direct sum decomposition of $D(\phi)_n$ as a $\pi_0(H)$-module. The face and degeneracy morphisms are, of course, those induced by the corresponding ones of H. In the above, we have used $N(D(\phi))_1$. If we write K_n for the simplicial subgroupoid, $Ker\phi_\bullet : H_\bullet \to \pi_0(H_\bullet)$, i.e. the kernel of the simplicial map to the constant simplicial groupoid, $\pi_0(H_\bullet)$, then K_n has a semidirect decomposition which is the same as that of H_n except that H_0 is replaced by d_0NH_1, so instead of the term $s_{n-1}\dots s_1s_0H_0$, it contains $s_{n-1}\dots s_1s_0d_0NH_1$. Using this subgroupoid, we find if $n \geq 1$

$$ND(\phi)_n \cong \frac{NH_n}{[NH_n, K_n]}$$

so the passage to $ND(\phi)$ merely kills the action of $Ker\phi$.

A *geometric* interpretation of this derived module construction is not obvious, however there is one which is related to the complex of chains on the universal cover of a CW-complex. The link is via ideas originally explored by Whitehead [17] and then considerably extended by Brown and Higgins [4]. Suppose G is an O-groupoid and $(M., \partial)$ is a complex of G-modules, then we form a crossed complex $\Delta(M_\bullet, \partial)$ by

$$\Delta(M_\bullet, \partial)_n = M_n \quad \text{if } n \geq 1$$

$$\Delta(M_\bullet, \partial)_0 = M_0 \rtimes G$$

where for $n > 1$, the boundary map

$$\partial_n : \Delta(M_\bullet, \partial)_n \to \Delta(M_\bullet, \partial)_{n-1}$$

is that of $M_\bullet$, whilst for $n = 1$, $\partial_1(m) = (\partial m, 1_p)$ if $m \in M_0(p)$. This defines a functor Δ from *Comp*(*G*–*Mod*) to *Crs*/*G*, i.e. crossed complexes augmented over the fixed G. This functor has a left adjoint:

If $(C_\bullet, \phi)$ is a crossed complex augmented over $\phi : C_0 \to G$, define

$$\zeta(C_\bullet, \phi)_n = C_n \quad \text{if } n \geq 2$$

$$\zeta(C_\bullet, \phi)_1 = C_1^{Ab}, \quad \text{the abelianization of } C_1,$$

$$\zeta(C_\bullet, \phi)_0 = D_\phi,$$

with the differential of $\zeta(C_\bullet, \phi)$ induced, in the obvious way, from that of $C_\bullet$. The proof that ζ is left adjoint to Δ should be fairly clear given our earlier 'recall' of the theory of derived modules, alternatively it can be found in [4].

Proposition 3.3. *If $H_\bullet$ is a simplicial groupoid, augmented via $\phi : H_\bullet \to \pi_0(H_\bullet)$, and $C(H_\bullet)$ is the associated crossed complex then $\zeta(C(H_\bullet), \phi) \cong ND(\phi)$.*

Proof. Consider the corresponding right adjoints and used the Dold–Kan theorem. □

Corollary 3.4. *For $n \geq 2$,*

$$C(H_\bullet) \cong ND(\phi)_n \cong \frac{NH_n}{[NH_n, K_n]}$$

where $K_n = (Ker\phi)$ as before.

Proof. By the previous discussion, ζ does nothing in dimensions greater than 1. □

Thus we have a second description of $C(H_\bullet)$ and hence of the verbal subgroupoid or at least of its Moore complex $N\mathscr{V}(H)_n \cong [NH_n, K_n]$.

In dimension 1, the verbal subgroupoid is generated by the Peiffer identities and hence can be conveniently written as

$$N\mathscr{V}(H)_1 = [Kerd_1, Kerd_0] = d_0(NH_2 \cap D_2)$$

(see [5]). Finally in dimension zero it is trivial as $C(H)_0 = NH_0 = H_0$. This gives our main theorem:

Theorem 3.5 (Generating words for the variety of crossed complexes). *The verbal simplicial subgroupoid of $G_\bullet$ corresponding to the variety of groupoid T-complexes is given by*

(i) $\mathscr{V}(G)_0$ *is trivial,*
(ii) $\mathscr{V}(G)_1$ *is generated by all* $[x, y], x \in NG_1, y \in Kerd_0$
(iii) $N\mathscr{V}(G)_n$ *is generated by all* $[x, y], x \in NG_n, y \in (Ker\phi)_n$, *for* $n \geq 2$.

Of course in $\mathscr{V}(G)_n$, we also have all of $s_\alpha N\mathscr{V}(G)_{n-r(\alpha)}$ and so to obtain a full description of a set of generators of $\mathscr{V}(G)_n$, one needs to include all images of generators of the lower terms in the Moore complex and their conjugates in G_n. To use Theorem 3.5 for calculations it will be necessary to look at the case of a free simplicial group or groupoid, $G_\bullet$ and to see if the analysis of Hall words can be adapted to this case.

Acknowledgements

The authors wish to thank the members of the Granada Algebra section and in particular A. Cegarra, M. Bullejos and P. Carrasco for useful conversations. The first author acknowledges the financial support of the University of Wales and of ERASMUS.

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